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LETTER TO THE EDITOR

de Sitter gauge invariance and the geometry of the Einstein–Cartan theory

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Abstract. A formulation of general relativity as a gauge theory of the de Sitter group SO(3, 2) is used to analyse the geometrical structure of the Einstein–Cartan theory. The SO(3, 2) symmetry must be spontaneously broken to the Lorentz group in order to reproduce the usual four-dimensional geometry of gravity. Special emphasis is placed upon the role of the Goldstone field of the symmetry breaking mechanism and also that of the original SO(3, 2) gauge fields. The latter are not directly identified with the gravitational *vierbein* and spin connection, but instead generate a kind of parallel transport known as development, which is the necessary construction to interpret the effects of space–time torsion and curvature.

The formal similarities between Yang–Mills gauge theories and general relativity have received considerable attention since the papers of Utiyama (1956), Kibble (1961) and Sciama (1962); a recent review of the attempts to describe gravity as a gauge theory has been given by Heyl *et al* (1976). A revival of interest in the connection between gravity and gauge theories has been stimulated by recent work in supergravity, which has been related to a gauge theory of the graded Poincaré (Chamseddine and West 1977) or graded de Sitter (MacDowell and Mansouri 1977) groups. The techniques from these two works have also been used in the analysis of O(2) extended supergravity (Townsend and van Nieuwenhuizen 1977) and in the construction of superconformal supergravity (Kaku *et al* 1978). While Chamseddine and West (1977) and MacDowell and Mansouri (1977) did not give actions that were strictly invariant under the indicated groups, it is possible to obtain such invariance by the introduction of a constrained field. This was shown by West (1978) for the case of pure gravity in a formulation that is invariant under the group SP(4), which is locally isomorphic to the de Sitter group SO(3, 2).

In this Letter we shall explain the geometrical significance of the formulation of gravity given by West (1978). The action given there is, in a new notation,

$$I = \int d^4x[my^A\epsilon_{ABCDE}\epsilon^{\mu\nu\rho\tau}R_{\mu\nu}^{BC}R_{\rho\tau}^{DE} + \lambda(y^Ay_A + m^{-2})] \tag{1}$$

where the capital indices run from 1 to 5, $\epsilon^{12345} = 1$, and the indices are raised and lowered with $\eta_{AB} = (1, 1, 1, -1, -1)$. The $R_{\mu\nu}^{AB}$ are the SO(3, 2) curvatures and y_A is an SO(3, 2) five-vector which may be eliminated to give a non-polynomial action containing solely the gauge fields.

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Since the field $y^A(x)$ is constrained to take its values in the surface $y^A y_A = -m^{-2}$, the SO(3, 2) symmetry of the theory is spontaneously broken in a way reminiscent of the nonlinear sigma model. There are no degrees of freedom associated with the field y^A, but it plays an important role in the geometrical structure of the theory. Since the SO(3, 2) symmetry is spontaneously broken, the field y^A gives rise to the Goldstone field of the theory upon passage to a nonlinear realisation (Coleman *et al* 1969, Callan *et al* 1969, Isham 1969, Salam and Strathdee 1969, Volkov 1973) of the SO(3, 2) symmetry. The Goldstone field $\zeta^a(x)$ is defined by

$$y^A(x) = \left\{ \sigma_\zeta[\exp(-i\zeta^a(x)P_a)] \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^{-1} \end{pmatrix} \right\}^A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^{-1} \end{pmatrix} + \begin{pmatrix} \zeta^a(x) \\ 0 \end{pmatrix} + O(\zeta^2) \quad (2)$$

where σ_5 denotes the five-vector representation of SO(3, 2), the lower case italic indices run from 1 to 4, and $P_a = mM_{a5}$ are the broken generators of the theory, scaled in the customary fashion by the inverse of the vacuum magnitude of the Higgs field. Equation (2) is the prototype for the passage to a nonlinearly transforming set of fields $\bar{\Psi}(x)$ from a field $\Psi(x)$ transforming according to a linear irreducible reprsentation σ of SO(3, 2):

$$\bar{\psi}(x) = \sigma[\exp(i\zeta^a(x)P_a)]\psi(x). \quad (3)$$

The importance of nonlinear realisations for gravity derives from the fact that the nonlinearly transforming fields mix only according to their stability subgroup indices, so in this case the Lorentz indices of fields will be respected by the full SO(3, 2) symmetry. Under the broken transformations generated by the P_a, the fields $\bar{\Psi}(x)$ transform independently according to their Lorentz indices with the group element $h_1(\zeta, \epsilon) \in$ SO(3, 1) defined by

$$\exp(-i\epsilon P)\exp(-i\zeta P) = \exp(-i\zeta' P)h_1(\zeta, \epsilon). \quad (4)$$

The Lorentz group element $h_1(\zeta, \epsilon)$ is a nonlinear function of ζ and ϵ whose exact expression can be worked out using the SO(3, 2) commutation algebra. A formalism for such calculations has been presented in Keck (1975) and Zumino (1977). Thus, under a P_a gauge transformation with parameters ϵ^a, $\zeta^a(x)$ transforms to $\zeta^{a'}(x)$ as given by equation (4), and $\bar{\Psi}(x)$ transforms to

$$\bar{\psi}'(x) = \sigma[h_1(\zeta, \epsilon)]\bar{\psi}(x). \quad (5)$$

The theory of nonlinear realisations allows us to define the true *vierbein* and spin connection of the Einstein–Cartan theory by passing from the original SO(3, 2) gauge fields $\omega_{\mu AB}$ to the redefined fields $\bar{\omega}^{ab}_\mu$ and $\bar{e}^a_\mu$ according to

$$\tfrac{1}{2}i\bar{\omega}^{ab}_\mu(x)M_{ab} - i\bar{e}_\mu{}^a(x)P_a = \exp(i\zeta(x)P)(\partial_\mu + \tfrac{1}{2}i\omega^{ab}_\mu(x)M_{ab} - ie^a_\mu(x)P_a)\exp(-i\zeta(x)P) \quad (6)$$

where the P_a gauge fields have been scaled to give the *vierbein* the correct dimension

$$\rho_{\mu a} = m^{-1}\omega_{\mu a 5}. \quad (7)$$

The identification of the gravitational *vierbein* and spin connection with redefined fields using the theory of nonlinear realisations was suggested by Volkov and Soroka (1973) in connection with the spontaneous symmetry breaking of supersymmetry. Later suggestions were also made by Gursey and Marcildon (1978) and Chang and Mansouri (1978). Volkov and Soroka did not interpret the coset parameters corresponding to the broken generators in the same way that we have, however. In their work, these coset parameters represented the points of space–time itself. In the present Letter, we have the Goldstone field $\zeta^a(x)$, which takes its values in an *internal* anti de Sitter space. The interpretation of the geometrical significance of this field is suggested by the fact that if $\bar{e}^a_\mu$ and $\bar{\omega}^{ab}_\mu$ are identified as the *vierbein* and spin connection, then the maximally symmetric solution to the field equations derived from equation (1) is an anti de Sitter space identical to the space that the $\zeta^a(x)$ take their values in. This may be seen by writing the action (equation (1)) in terms of the redefined fields $\bar{e}^a_\mu$ and $\bar{\omega}^{ab}_\mu$, with the result

$$I=\int \mathrm{d}^4x\epsilon_{abcd}\epsilon^{\mu\nu\rho\tau}\bar{R}^{ab}_{\mu\nu}\bar{R}^{cd}_{\rho\tau} \tag{8}$$

where the de Sitter curvatures $\bar{R}^{ab}_{\mu\nu}$ are given by the same expressions as $R^{ab}_{\mu\nu}$, but with $\bar{e}^a_\mu$ and $\bar{\omega}^{ab}_\mu$ in place of e^a_μ and ω^{ab}_μ. The form (equation (8)) of the gravitational action is the same as was given by MacDowell and Mansouri (1977), but expressed here in terms of the barred fields to retain the full SO(3, 2) invariance. The action (equation (8)) may be expanded into three terms, which are the Gauss–Bonnet topological invariant, the Hilbert action and a cosmological constant. The cosmological constant is proportional to m^4, so its value is determined by the strength of the symmetry breaking.

Since the maximally symmetric solution to the equations derived from equation (8) may be identified with the space given by $y^Ay_A=-m^2$, the natural interpretation of the Goldstone field $\zeta^a(x)$ is as the coordinates of a point associated to x^μ in a local copy of the vacuum. This is reminiscent of the standard inertial frames of general relativity, and in fact the analogy can be carried out in complete detail. We shall see that the internal anti de Sitter space at each point x^μ can be used to construct a local coordinate system in the vicinity of x^μ.

In order to explain more fully the geometrical role of the Goldstone field $\zeta^a(x)$ and of the space in which it lies, we must direct our attention to the operation of parallel transport. In complete agreement with the usual situation in the Einstein–Cartan theory, the usual notion of parallel transport is generated by the covariant derivative $\bar{D}_\mu$, defined by

$$\bar{D}_\mu=\partial_\mu+\tfrac{1}{2}\mathrm{i}\bar{\omega}^{ab}_\mu M_{ab}. \tag{9}$$

The non-linear field $\bar{\omega}^{ab}_\mu$ transforms correctly to make this a covariant derivative when acting on non-linearly transforming fields such as $\bar{\Psi}(x)$. Fields with 'world' indices may be parallelly transported using the connection

$$\Gamma^\lambda_{\mu\nu}=\bar{e}^\lambda{}_a(\partial_\mu\bar{e}_\nu{}^a+\bar{\omega}^a_{\mu b}\bar{e}_\nu{}^b) \tag{10}$$

where $\bar{e}^\lambda{}_a$ is the inverse of $\bar{e}_a{}^\lambda$.

Under parallel transport with equation (9), there is little indication of the role of the original SO(3, 2) invariance of the theory, for all the fields have Lorentz indices and the remaining invariance under the P_a transformations is automaticallly achieved through barring all the fields. The original linear SO(3, 2) gauge fields do still have an important role to play, however, in the process of development, which is defined by another differential operator Δ_μ that is given by

$$\Delta_\mu = \partial_\mu + \tfrac{1}{2}\mathrm{i}\omega_\mu^{ab} M_{ab} - \mathrm{i}e_\mu{}^a P_a. \qquad (11)$$

In equation (11), the elements M_{ab} and P_a of the SO(3, 2) Lie algebra are considered to be operators that act upon non-linear fields according to their transformation type. Thus, M_{ab} operates linearly according to the Lorentz indices, but P_a gives rise to an infinitesimal transformation that is non-linear in $\zeta^a(x)$, as can be seen from equations (4) and (5). For example,

$$-\mathrm{i}\epsilon^b P_b \bar{V}^a = [h_1(\zeta, \epsilon) - \mathbb{1}]^a_b \bar{V}^b. \qquad (12)$$

The operator Δ_μ is covariant in the following sense: when $(\mathbb{1} + \mathrm{d}x^\mu \Delta_\mu)$ is applied both to $\zeta^a(x)$ and to some vector field $\bar{V}^a(x)$ to produce $\zeta^a_*(x)$ and $\bar{V}^a_*(x)$, then $\zeta^a_*(x)$ and $\bar{V}^a_*(x)$ transform using the same functions of ζ^a_* and the transformation parameters as the functions of ζ^a given in equations (4) and (5), keeping only up to first order terms in $\mathrm{d}x^\mu$.

The process of development generated by Δ_μ may be understood by re-expressing the action of Δ_μ on $\bar{V}^a$ using the following:

$$\Delta_\mu \bar{V}^a = (\bar{D}_\mu + \mathrm{i}\bar{e}^b_\mu P_b)\bar{V}^a. \qquad (13)$$

The first term just generates ordinary parallel transport across space–time, as we have said. The second term generates parallel transport in the internal anti de Sitter space. More precisely, for an infinitesimal development a distance $\mathrm{d}x^\mu$, the quantity $\bar{V}^a_*$ that we have referred to above is the result of parallelly transporting $\bar{V}^a(x + \mathrm{d}x)$ to x^μ using $\bar{D}_\mu$, and then parallelly transporting it in the internal space away from $\zeta^a(x)$ to the point ζ^a_*. Note that the sign on the second term in equation (13) is different from equation (11), as it must be to achieve this. The situation requires a clear understanding of the true role of the *vierbein* $\bar{e}^a_\mu(x)$. The *vierbein* is the matrix of a map between the tangent space to space–time at x^μ and the tangent space to the internal anti de Sitter space at the point $\zeta^a(x)$. Thus, development takes advantage of the possibility of moving to points other than $\zeta^a(x)$ in the internal space associated to the point x^μ.

Development along finite curves in space–time gives rise to image curves in the internal space. Vector fields defined along the curves in space–time are mapped into image vector fields along the image curves. The identification of points in space–time with points in the internal space is not unique, however. The non-integrability of development may be investigated in the usual fashion by considering an infinitesimal closed curve in space–time and developing it and a vector situated at its starting point x_0^μ into the internal space associated with the point x_0^μ. Keeping only up to terms of second order in the displacement $(x^\mu - x_0^\mu)$, the result of developing a vector $\bar{V}^a(x_0)$ around the curve is $\bar{V}^a_{*(2)}(x_0; x_0)$, which is given by

$$\bar{V}^a_{*(2)}(x_0; x_0) - \bar{V}^a(x_0) = \tfrac{1}{2}[\tfrac{1}{2}\mathrm{i}R^{bc}_{\mu\nu}(x_0)M_{bc} - \mathrm{i}R^b_{\mu\nu}(x_0)P_b]\bar{V}^a(x_0) \oint x^\mu\, \mathrm{d}x^\nu. \qquad (14)$$

Analogously to equation (13), this may be re-expressed as

$$\bar{V}^a_{*(2)}(x_0; x_0) - \bar{V}^a(x_0) = \tfrac{1}{2}[\tfrac{1}{2}\mathrm{i}\bar{R}^{bc}_{\mu\nu}(x_0)M_{bc} + \mathrm{i}\bar{R}^b_{\mu\nu}(x_0)P_b]\bar{V}^b(x_0) \oint x^\mu\, \mathrm{d}x^\nu. \qquad (15)$$

The second term in equation (15) shows that, after development around a closed curve, in order to return $\bar{V}^a_*$ to its original value an internal parallel transport is necessary. Thus the end of the image curve is not at the same point as its beginning, and so the torsion $\bar{R}^a_{\mu\nu}$ indicates the gap in an image curve corresponding to an infinitesimal closed curve in space–time. The first term in equation (15) gives the rotation of $\bar{V}^a_*(x_0; x_0)$ with respect to $\bar{V}^a(x_0)$, exclusive of that induced by parallel transport across the gap in the image curve. This rotation is given by the difference between two terms, as may be seen by expanding $\bar{R}^{ab}_{\mu\nu}$:

$$\bar{R}^{ab}_{\mu\nu} = \bar{B}^{ab}_{\mu\nu} + m^2(\bar{e}^a_\mu \bar{e}^b_\nu - \bar{e}^a_\nu \bar{e}^b_\mu) \tag{16}$$

where $\bar{B}^{ab}_{\mu\nu}$ is just the usual Lorentz curvature expressed in terms of $\bar{\omega}^{ab}_\mu$. The remainder of equation (16) is the negative of the Lorentz curvature tensor of anti de Sitter space. The de Sitter curvature $\bar{R}^{ab}_{\mu\nu}$ is thus the difference between the usual curvature of space–time and the curvature of the internal space. This is as it should be, because development involves parallel transport both across space–time and within the internal space, in opposite directions: first in from $x^\mu + dx^\mu$ to x^μ in space–time, then out in the internal space from $\zeta^a(x)$ to ζ^a_*.

The process of development is particularly simple when space–time is in its vacuum state, for then $R^{AB}_{\mu\nu} = 0$, and one can pick a gauge where $\omega^{AB}_\mu = 0$. Then the *vierbein* $\bar{e}^a_\mu$ and spin connection $\bar{\omega}^{ab}_\mu$ are entirely given by functions of $\zeta^a(x)$ and its derivatives. In this case, we also have $\Delta_\mu = \partial_\mu$, so development of non-linear fields $\bar{V}^a$ and the Goldstone field ζ^a involves doing nothing at all to the components of these fields. Thus, picking some point x^μ_0, the surrounding points of space–time may be associated with points in the internal space at x^μ_0 by simply associating x^μ with the point $\zeta^a(x)$. The internal space at x^μ_0 may therefore be considered as a map of the surrounding space–time.

In general space–times, the identification of points in space–time with the points of the internal space at x^μ_0 cannot be done unambiguously. Thus it is necessary to decide upon some definite procedure to establish the mapping of space–time. The natural way to do this is to map along autoparallels in space–time passing through x^μ_0, satisfying

$$\frac{d^2x^\mu(t)}{dt^2} + \Gamma^\mu_{\rho\sigma}(x(t))\frac{dx^\rho(t)}{dt}\frac{dx^\sigma(t)}{dt} = 0. \tag{17}$$

Autoparallels are mapped into geodesic image curves in the internal space at x^μ_0. In this way, we can establish a local coordinate system in the vicinity of x^μ_0. If gauges are chosen for the P_a transformations such that $\zeta^a(x) = 0$, then this coordinate system will be a normal coordinate system, inheriting this property from the parametrisation (equation (2)) of the internal space at its origin.

We have seen above that the internal anti de Sitter spaces are the analogues of the local inertial frames of the standard formulation of general relativity, when used to establish local coordinate systems using development. The process of development that we have derived is a purely group theoretic notion of a form of parallel transport of the Goldstone and vector fields. It has been given its name because it generates image curves and vector fields which can be shown to be identical to those obtained by a generalisation to our situation with internal anti de Sitter spaces of a purely geometric construction in differential geometry known as development into the flat affine tangent space of a differentiable manifold, as is discussed for example by Kobayashi and Nomizu (1963). Using it, we have achieved a full agreement of the Yang–Mills gauge theoretic and the geometrical aspects of the Einstein–Cartan theory.

The results presented in this Letter will be discussed in fuller detail in a separate publication.

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